SOLUTION OF THE VARIATIONAL PROBLEM OF CONSTRUCTING THE CONTOUR OF A COMPOUND NOZZLE

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The variational problem is solved of constructing the contour of the supersonic part of an optimal compound nozzle intended to work in two essentially different regimes. Thus the complete nozzle works in a regime that is characterized by large overexpansion of pressure. In a regime with smaller overexpansion the final section of the nozzle is retracted (or jettisoned). There are given the maximum permissible length of the full nozzle, the back pressure determining each regime, and the probabilities of using the full nozzle and the partial one. Optimization is carried out for the average thrust.

Necessary conditions are obtained that permit constructing an optimal contour, and a corresponding numerical algorithm is developed based on these conditions. Examples are given of optimal compound nozzles constructed with the use of this algorithm, and comparison is made with the optimal continuous nozzles calculated for the average back pressure. An analysis is made of the evolution of the shape of the optimum compound nozzle in the whole range of possible values of the maximum allowable length.

The question of the possibility of applying a compound nozzle was considered in [1, 2]. The profiling of such a nozzle cannot be carried out according to existing solutions [3–5], and obtaining the necessary extremal conditions requires the application of the general method of Lagrange multipliers. In the solution of variational problems in gas dynamics this method was first applied by Guderley and Armitage [6, 7] and independently, though somewhat later, by Sirazetdinov [8].

1. We consider a plane ($\nu = 0$) or axisymmetric ($\nu = 1$) nozzle (Fig. 1), of which the final section db can be separated from the initial section ad. We will call such a nozzle compound. Let the gas flow from left to right, and the axes of a rectangular coordinate system xy, which in the axisymmetric case lies in the meridional flow plane, be placed so that the initial point a of the nozzle contour to be found lies on the y-axis. The contour to the left of a is regarded as given, where in the general case point a is a corner (the direction of the contour sought to the right of a does not necessarily agree with the direction of the contour given to the left of a). We restrict ourselves to the case when shock waves are absent from the part of the region of influence of the desired contour lying to the left of the characteristic hb.

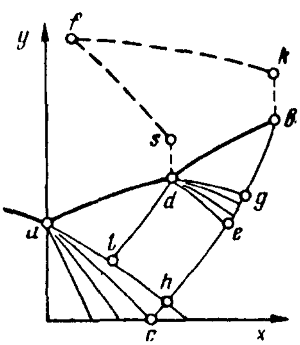

Fig. 1

We assume that the gas is inviscid and non-heat-conducting, and its entropy and stagnation enthalpy at $x = 0$ are given and constant across the section. Under these assumptions these quantities remain constant everywhere to the left of hb. Therefore the pressure p, density ρ, speed of sound c and other thermodynamic variables are functions of the speed w, and to determine the flow variables it suffices to use the equations of irrotationality and continuity

$$L_1 \equiv \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \qquad L_2 \equiv \frac{\partial y^\nu \rho u}{\partial x} + \frac{\partial y^\nu \rho v}{\partial y} = 0 \tag{1.1}$$

where u and v are the projections of the velocity vector onto the x- and y-axes.

If the magnitude of the corner angle at point a exceeds a certain value, which is determined by the shape of the contour for $x < 0$, the flow in the transonic region does not depend on the shape of the contour for $x > 0$ nor, in particular, on the angle ϑ_a of inclination of the contour of the wall to the x-axis to the right of a. In this case the magnitude of ϑ_a affects only the extent of the expansion fan springing from a, that is, the location of the characteristic ah of the second family, which bounds this fan on the right. Therefore the variables on some "inner" characteristic of the fan, for example on ac, can be regarded as given. In this connection the flow in region G, bounded by the characteristics ac and cb and the contour adb, is determined (for a given characteristic ac and flow variables on it) by equations (1.1) and the condition of no flow

$$L \equiv \xi' - u / v = 0 \tag{1.2}$$

through the wall of the nozzle. In (1.2) and henceforth a prime indicates the total derivative with respect to y along the contour adb, and $x = \xi(y)$ is the equation of this contour.

Together with the sections ad and db, the nozzle has the afterbody portions, whose contours bkf and dsf are not exposed to the gas flow and are shown by dashed lines in Fig. 1. The pressures p^+ and $p^{+\circ}$ that act on bkf and dsf, respectively, are given constants, characterizing the working regime of the full nozzle and the partial one. In the general case point d, like point a, can be a corner point. Such a situation is shown in Fig. 1, where de and dg are characteristics of the second family bounding the corresponding expansion fan.

By virtue of the assumptions made, the thrusts χ and χ° of the complete and the truncated nozzles are, to within an additional positive factor that is not essential for what follows, equal to

$$\chi = \int_a^b p y^\nu dy - y_b^{1+\nu} \frac{p^+}{1+\nu}, \qquad \chi^\circ = \int_a^d p y^\nu dy - y_d^{1+\nu} \frac{p^{+\circ}}{1+\nu} \tag{1.3}$$

Here and subsequently the subscripts b, d,... indicate variables at the corresponding points.

We formulate the variational problem. Let there be given the maximum allowable length X of the complete nozzle, the pressures p^+ and $p^{+\circ}$, the positive numbers n and n°, the enthalpy and entropy of the gas at entry into the nozzle, and the shape of its subsonic part (as mentioned above, in this case the flow can be regarded as given to the left of the characteristic ac). It is required to construct a contour adb, that is, to find the relationship $x = \xi(y)$, where $0 \leqslant \xi(y) \leqslant X$, and the coordinates of points d and b, such that the compound nozzle realizes the maximum "average" thrust

$$\chi_\Sigma = n\chi + n^\circ \chi^\circ \tag{1.4}$$

The coefficients n and n° in (1.4), which are determined by the purposes of the nozzle, are the probabilities of using the complete nozzle and the partial one; and it is convenient to normalize so that $n + n^\circ = 1$. Then (1.4) together with (1.3) gives

$$\chi_\Sigma = \int_a^d py^\nu dy + n\int_d^b py^\nu dy - ny_b^{1+\nu}\frac{p^+}{1+\nu} - (1-n)\,y_d^{1+\nu}\frac{p^{+\circ}}{1+\nu} \qquad (1.5)$$

For $n = 1$ we have $n^\circ = 0$, and thus only the complete nozzle is used ($\chi_\Sigma = \chi$). For $n < 1$ the contributions of the initial and final portions of the contour to the functional (1.5) are different. This also serves to explain why in the general case the optimal contour has a corner at point d. As one more condition on the problem we may require that the length of the truncated nozzle be determined by the abscissa of point d, that is, that the condition $x \leqslant x_d$ be satisfied on fd.

It is convenient to regard the variables in (1.1)–(1.5) as dimensionless. In reducing to dimensionless form it is convenient to take as the characteristic length, speed, and density (l_*, w_*, ρ_*) the ordinate of point a and the critical speed and density of the flow. Nondimensionalization is achieved by referring quantities of dimension length to l_*, speed to w_*, density ρ_*, pressure $\rho_* w_*^2$ and thrust to $\rho_* w_*^2 l_*^{1+\nu}$. In the plane case χ, χ°, and χ_Σ in (1.3)–(1.5) are quantities per unit width of the nozzle (in the direction perpendicular to the xy-plane).

2. To solve the formulated variational problem, we construct the auxiliary functional

$$J = \chi_\Sigma + \int_a^b \alpha L dy + \iint_G (\mu_1 L_1 + \mu_2 L_2)\,dx\,dy \qquad (2.1)$$

where $\alpha = \alpha(y)$ and $\mu_i = \mu_i(x, y)$ are variable Lagrange multipliers. By virtue of Eqs. (1.1) and (1.2), for an admissible variation, when the velocity components u and v and also the density and pressure, being known functions of u and v, satisfy the equations and boundary conditions of the problem, the first variation δJ coincides with the first variation $\delta\chi_\Sigma$ of the initial functional.

In finding δJ it must be kept in mind that for small variations of the contour adb the gas variables change only in the subregion G° of the region G that lies to the right of ah, and also a displacement of this characteristic occurs. The variations of variables to the left of ah are equal to zero. Considering this, and using the equations of motion (1.1), it is possible to show that although the variations δu and δv are different from zero on ah (by virtue of the displacement of ah due to change in the angle of discontinuity at point a), their combinations appearing in δJ for the variation of the integral over G vanish on ah. The calculation of the contribution to δJ associated with variation of the coordinates of point d is carried out just as in [9]. Also, discontinuities in the factors μ_1 and μ_2 are admitted, which can occur only on characteristics [9, 10].

After the calculation of δJ the coefficients in front of all the variations, which are different from the variations of the coordinates of the contour adb, can be set equal to zero by choice of the Lagrange multipliers α, μ_1 and μ_2. As a result is obtained the "associated" problem for the determination of α on adb and the multipliers μ_1 and μ_2 in region G°. Thus in the subregion of their continuity, μ_1 and μ_2 must satisfy the following system of partial differential equations

$$\frac{\partial \mu_1}{\partial y} + y^\nu (\rho u)_u \frac{\partial \mu_2}{\partial x} + y^\nu \rho_u v \frac{\partial \mu_2}{\partial y} = 0 \qquad (2.2)$$

$$\frac{\partial \mu_1}{\partial x} - y^\nu \rho_v u \frac{\partial \mu_2}{\partial x} - y^\nu (\rho v)_v \frac{\partial \mu_2}{\partial y} = 0$$

This system has for $w > c$ two families of real characteristics, which coincide with the characteristics of the equations of motion (1.1) and on which

$$d\mu_1 \mp y^\nu \rho\beta \, d\mu_2 = 0 \qquad (\beta = \sqrt{M^2 - 1}) \qquad (2.3)$$

Here and subsequently the upper (lower) sign corresponds to characteristics of the first (second) family, and $M = w / c$ is the Mach number. The differentials $d\mu_1$ and $d\mu_2$ in (2.3) are taken along the characteristics.

On characteristics that lie in G° and are lines of discontinuity of the Lagrange multipliers, the jumps in μ_1 and μ_2 satisfy the relation

$$[\mu_1] \pm y^\nu \rho\beta [\mu_2] = 0 \qquad (2.4)$$

where $[\mu_i]$ is the difference in the values of μ_i to the right and the left of the discontinuity.

The boundary conditions associated with the problem for μ_i are formulated on the nozzle wall and on the final characteristic hb, and have the form

$$\mu_1 = y^\nu \rho v \quad \text{on } ad, \qquad \mu_1 = y^\nu \rho v n \quad \text{on } db, \qquad \mu_1 + y^\nu \rho\beta\mu_2 = 0 \quad \text{on } hb \qquad (2.5)$$

Finally, the Lagrange multiplier α on ad and db is determined so that

$$\alpha + y^\nu \rho v \mu_2 = 0 \qquad (2.6)$$

Use of the third condition from (2.5) permits integration of the equation from (2.3) corresponding to characteristics of the first family, and thus finding μ_1 and μ_2 on hb in terms of y and the flow parameters. The appropriate equations have the form

$$\mu_1 = C (y^\nu \rho\beta)^{1/2}, \quad \mu_2 = -C (y^\nu \rho\beta)^{-1/2} \quad \text{on } hb \qquad (2.7)$$

Here C is a constant that is determined, for example, by comparing the values of μ_{1b} obtained from (2.5) and (2.7).

For an arbitrary contour adb, for which the flow takes place without formation of a shock wave in G°, the equations and boundary conditions (2.2)–(2.7) permit solution of the associated problem and finding, in particular, the values of the Lagrange multipliers on the contour adb. Here it can be shown that in the case illustrated in Fig. 1 the line of discontinuity of the multipliers μ_1 and μ_2 is the characteristic dl of the first family passing through point d. After the choice of the Lagrange multipliers the expression for $\delta\chi_\Sigma = \delta J$ takes the form

$$\delta\chi_\Sigma = A\Delta y_d + B\Delta x_d - n y_b{}^\nu \left(p^+ - p + \frac{\rho}{\beta} uv\right)_b \Delta y_b +$$
$$+ \left(y^\nu \rho v^2 \frac{n}{\beta}\right)_b \Delta x_b + \int_a^d \rho v y^\nu (\mu_2 - u)' \, \delta\xi \, dy +$$
$$+ \int_d^b \rho v y^\nu (\mu_2 - nu)' \delta\xi \, dy \qquad (2.8)$$

$$A = y_d^{\nu}(p_- - np_+ - n^\circ p^{+\circ})_d - [\mu_2 y^\nu \rho u]_d - \int_{d-}^{d+} \{\mu_1 dv - \mu_2 y^\nu d(\rho u)\}$$

$$B = [\mu_2 y^\nu \rho v]_d - \int_{d-}^{d+} \{\mu_1 du + \mu_2 y^\nu d(\rho v)\}$$

Here the integrals at point d are taken through the whole fan of the expansion wave ; the subscripts "minus" ("plus") are associated with parameters on the wall before (after) the point of discontinuity; $[\varphi] = \varphi_+ - \varphi_-$; and $\delta\xi$ designates the variation of the abscissa of the wall (for fixed y), and Δx and Δy the increments in the coordinates of the corresponding point.

If the contour adb is optimal, then for an admissible variation the variation $\delta\chi_\Sigma$ is nonpositive. For $x_d < x_b$ this, and consideration of limitations on the length of the complete nozzle, lead to the following conditions for determining the shape of the optimal contour :

$$(\mu_2 - u)' = 0 \text{ on } ad, \qquad (\mu_2 - nu)' = 0 \text{ on } db$$
$$A = 0, \quad B = 0 \quad \text{at point } d \tag{2.9}$$
$$(p^+ - p + \rho uv\beta^{-1})_b = 0, \qquad (y^\nu \rho v^2 n \beta^{-1})_b \geqslant 0$$

Here the third and fourth, and the fifth and sixth conditions, respectively, determine the ordinate and abscissa of points d and b, where fulfillment of the inequality in the latter conditions indicates that the length of the whole nozzle is equal to the maximum permissible.

In the specified range of the parameters X, n, p^+, and $p^{+\circ}$, in particular if $p^{+\circ} \simeq p^+$ the optimal is not a compound but a simple nozzle when $x_d \equiv x_b$. It can be shown that in such a case the ordinate of the end point is determined by the next to the last equation (2.9) with p^+ replaced by the mean counter-pressure $p_\Sigma^+ = np^+ + n^\circ p^{+\circ}$. The condition that the maximum χ_Σ be achieved by a continuous nozzle has the form

$$\{(\rho v^2\beta^{-1})_- - n(\rho v^2\beta^{-1})_+\}_b \geqslant 0 \tag{2.10}$$

In the case when this inequality is satisfied, introduction of an infinitely small removable end part leads to reduction in the thrust of the nozzle. The slope ζ_+ of that end part, where $\zeta = v / u$, is chosen optimal, that is, such that the parameters (with subscript "plus") that are obtained on the wall after turning from $\zeta = \zeta_- \equiv \zeta_b$ to $\zeta = \zeta_+$ satisfy the next to the last condition (2.9). Since $p^+ < p_\Sigma^+$, then $\zeta_+ > \zeta_-$. The "minus" subscript in (2.10) is assigned to parameters on the wall of a continuous nozzle. The condition (2.10) can be obtained by direct variation of the final element of the nozzle, as well as from (2.8). Here it is necessary to consider the connection between the admissible increases in the coordinates of points b and d for $x_d = x_b$ and the fact that here $\Delta x_d \leqslant 0$.

The process of constructing an optimal nozzle can be simplified in the following way. If the segments ad and db are optimal, then according to (2.5) and (2.9)

$$\mu_1 = y^\nu \rho v, \quad \mu_2 = u + C_1 \quad \text{on } ad, \qquad \mu_1 = y^\nu \rho v n, \quad \mu_2 = nu + C_2 \quad \text{on } db \tag{2.11}$$

where C_1 and C_2 are constants. Construction of the segments ad and db of the optimal contour by virtue of the solution of the corresponding Goursat problem is equivalent to determining the "optimal" characteristics ld and bg. The change to those characteristics is realized thanks to the fact, that just as in [11], the first pair of equations (2.11) gives the solution of the Cauchy problem for the system (2.2) with initial conditions (2.11) on ad in the whole triangle adl. In an analogous way the second pair of equations in (2.11) gives the solution of the corresponding Cauchy problem in dbg. As a result it is necessary to determine the Lagrange multipliers in the solution of the associated problem only in the quadrangle $ldgh$. Here the boundary conditions for μ_1 and μ_2 are imposed on dg and gh.

The optimal characteristic ld is determined by the equation

$$E \equiv \mu_1 / y^\nu \rho\beta + \mu_2 - u - v\beta^{-1} - C_1 = 0 \quad \text{on } ld \tag{2.12}$$

which replaces the first of the conditions in (2.9) on ad. The Lagrange multipliers appearing in (2.12) are taken on the right side of the characteristic ld. In an analogous way the condition determining the optimal characteristic bg has the form

$$v + \beta (u + C_2) = 0 \quad \text{on } bg \tag{2.13}$$

The values of μ_1 and μ_2 on dg, required for solving the associated problem in $ldgh$ are given by the equations

$$\mu_1 = y^\nu \rho v n, \qquad \mu_2 = nu + C_2 \quad \text{on } dg \tag{2.14}$$

These equations, when written at point g, together with (2.7) permit the constants C and C_2 to be expressed in terms of n and the flow parameters at that point.

We note that condition (2.13) on bg reduces to the known condition of optimality that is obtained on the closing characteristic in problems of optimization of a continuous nozzle [3, 4]. The given result is natural since the closing section works only in one regime, and by virtue of the supersonic character of the flow changing its shape does not affect the initial section ad.

The equations and boundary conditions obtained above form the basis of a numerical algorithm for constructing the sections ad and db of the optimal nozzle in gas flow and, in particular, for determining the coordinates of points b and d. Here the outline of the afterbody section is merely required to join the initial point f with points b and d, and the length of the afterbody section must not exceed X for the entire nozzle and x_d for its section. Therefore, although the shape of the contour of the afterbody sections is arbitrary in a given case, they may contain butt ends bk and ds, where $x \equiv X$ and $x \equiv x_d$ respectively [12]. The forces acting on the afterbody sections do not depend on their configuration. Such a statement holds only in the absence of external flow. If in the regime of operation of the entire nozzle the afterbody section is in a supersonic stream, then its construction and the determination of the coordinates of point b are carried out as in [13].

3. In the variational problem under consideration X, n, p^+ and $p^{+\circ}$ are given. The numerical algorithm for constructing the optimal contour turns out to be more simple for the "inverse problem". For that, instead of the indicated values, the following ones are given: the corner angle of the contour at the initial point (consequently the closing characteristic of the expansion fan springing from point a), the coordinates x_l and x_h which determine the location of points l and h on the closing characteristic of the first fan, and the corner angle of the contour at point d. Here, as was mentioned above, construction of the initial section of a compound nozzle is equivalent to the construction of the optimal characteristic ld, that is the determination on it, for example, of the relationship $\zeta = \zeta(\psi)$. The stream function ψ is introduced as usual such that on the axis $\psi = 0$ and on the wall $\psi = 1$.

The optimal distribution $\zeta(\psi)$ on ld must satisfy the condition (2.12), in which μ_1 and μ_2 are found from the solution of a Goursat problem in $ldgh$ with boundary conditions (2.14) on dg and (2.7) on hg. Satisfying the condition (2.12) at point l provides the choice of the constant C_1. A different distribution of ζ from the optimal means a violation of condition (2.12), that is, the equality $E = 0$ holds at a point on the characteristic ld different from l. This property is used to organize an iteration process for the determination of the optimal distribution of ζ on ld. The iteration was carried out according to the scheme

$$\zeta_n^j = \zeta_n^{j-1} + \varepsilon_n^j E_n^j \tag{3.1}$$

which is analogous to the scheme employed in [14, 15]. In (3.1) the subscript gives the number of the point on ld and the superscript the number of iteration, and the ε_n^j are constants, where $|\varepsilon_n^j| < 1$. The quantities ε_n^j in these bounds can depend on the number of the point and the number of iteration. Since a given point l the value ψ_l is known, and $\psi_d = 1$, it is convenient to arrange the points on ld so that fixed subscripts in (3.1) correspond to fixed ψ.

In each iteration the relation $\zeta(\psi)$ found from (3,1) together with the equations of the characteristic of the first family completely determines the characteristic ld. Then from the solution of the Goursat problem for the equations of flow in the quadrangle $ldeh$ with known parameters on ld and lh and the subsequent calculation of the expansion fan edg, the flow is found in the whole quadrangle $ldgh$. This in its turn permits solution of the Goursat problem for μ_1 and μ_2 and making a new iteration according to (3.1). When the condition $E_n = 0$ is satisfied with given accuracy at all points of the characteristic ld, the optimal characteristic bg is constructed. For this, integration of the equations of the characteristic of the first family is carried out from $\psi = \psi_g$ to $\psi = 1$, with Eq. (2.13) taken into account.

In case the optimal turns out to be a nozzle of the kind considered (Fig. 1), $v_b > 0$ and in the last condition of (2.9) the inequality holds, that is $X = x_b$. The pressure p^+ characterizing the working regime of the complete nozzle is, in the given "inverse" approach, found (after determination of the parameters at point b) from the next to the last condition of (2.9). Finally, n and $p^{+\circ}$ are selected so as to satisfy the third and fourth conditions of (2.9). The latter, with regard to the expressions for A and B and the solutions (2.11), have the form

$$A y_d^{-\nu} \equiv \Big\{p_- - np_+ - n^\circ p^{+\circ} - (nu_+ + C_2)(\rho u)_+ + $$

$$+ (u_- + C_1)(\rho u)_-\Big\}_d - \int\limits_{d-}^{d+} \{\mu_1 y^{-\nu} dv - \mu_2 d(\rho u)\} = 0$$

$$B y_d^{-\nu} \equiv \{(nu_+ + C_2)(\rho v)_+ - (u_- + C_1)(\rho v)_-\}_d -$$

$$- \int\limits_{d-}^{d+} \{\mu_1 y^{-\nu} du + \mu_2 d(\rho v)\} = 0 \tag{3.2}$$

If the n, p^+, $p^{+\circ}$, p_{b-} and p_{d-} found as the result of solution of the "inverse" problem satisfy the inequalities

$$0 \leqslant n \leqslant 1, \quad 0 \leqslant p^+ \leqslant p_{b-}, \quad 0 \leqslant p^{+\circ} \leqslant p_{d-}$$

then the values obtained for X, n, p^+ and $p^{+\circ}$ can be regarded as the data for some original variational problem. We note that the construction of a continuous optimum nozzle is also based on the solution of an inverse problem.

In the reasoning given above the fact was ignored that in Eqs. (2.14) for μ_1 and μ_2 on dg there appears n, which is known only after construction of the characteristic ld. This discrepancy is, for $n > 0$, eliminated by setting

$$\mu_i^\circ = \mu_i / n, \quad C^\circ = C / n, \quad C_2^\circ = C_2' / n$$

The equations and boundary conditions for μ_i° in the quadrangle $ldgh$ are obtained from (2.2), (2.3), (2.7) and (2.14) by replacing μ_i, C and C_2, by μ_i°, C° and C_2°, and n by unity. At the same time μ_i and C_2 in (2.12) and (3.2) must be replaced by $n\mu_i^\circ$ and nC_2°. This permits elimination of n from the equations and boundary conditions of the associated problem. Here n and C_1 in each iteration are found from the second equation of (3.2), which is linear with respect to n and C_1, and from the equation $E_l = 0$.

The speed of convergence of the iteration process (3.1) depends on the choice of the initial distribution of ζ on ld. For a small final section db it is natural to take the distribution corresponding to the optimal for the continuous nozzle [3, 4]. Then each new construction of an optimal distribution of ζ on ld is taken as the initial for constructing a nozzle with a longer final section, a larger corner angle in the wall, etc. The iteration of ζ on ld is carried out as long as $| E_n / \zeta_n |$ everywhere on ld becomes less than some sufficiently small value. Using the expression for $\delta\chi_\Sigma$, we can show that the error $\Delta\chi_\Sigma$ in the thrust of a nozzle constructed in this way is a quantity of order $E_{n\max}^2$, where $E_{n\max}$ is the maximum disparity on ld.

4. The algorithm given above was applied to the construction of a large number of optimal contours. Axisymmetric nozzles were considered with a plane transition surface, departures from which were treated according to [16]. The gas was assumed perfect with adiabatic exponent $\varkappa = 1.4$. Iterations were carried out until the condition $| E_n/\zeta_n | < 0.01$ was satisfied at every point of ld; in all cases considered, from two to seven iterations were required.

The optimal distributions of ζ on ld for some nozzles that are obtained for $\zeta_a = 0.221$, $\Delta\zeta_d \equiv \zeta_{d+} - \zeta_{d-} = 0.15$ and a fixed point l are shown in Fig. 2. Curves 1–6 correspond to nozzles that are optimal for the following values of X, n, p^+ and $p^{+\circ}$ and have the geometric properties shown in Table 1

Table 1

	1	2	3	4	5	6
X	= 2.91	3.57	4.34	5.25	6.29	7.51
n	= 0.35	0.37	0.39	0.42	0.45	0.49
$p^+ \times 10^2$	= 0.48	0.67	0.72	0.69	0.59	0.44
$p^{+\circ} \times 10$	= 1.16	1.10	1.03	0.95	0.86	0.75
y_b	= 1.62	1.79	1.99	2.20	2.45	2.74
x_d	= 2.42	2.44	2.46	2.48	2.51	2.55
y_d	= 1.49	1.51	1.52	1.55	1.57	1.61
$\zeta_{d-} \times 10$	= 1.34	1.41	1.48	1.57	1.67	1.81
$\zeta_b \times 10$	= 2.52	2.24	1.99	1.78	1.62	1.52

The zero Curve in Fig. 2 gives the distribution of ζ on the closing characteristic of the optimal continuous nozzle ($X \approx 2.4$) that passes through the same point l; the axis of abscissas gives $\Delta\psi = (\psi - \psi_l) / (1 - \psi_l)$.

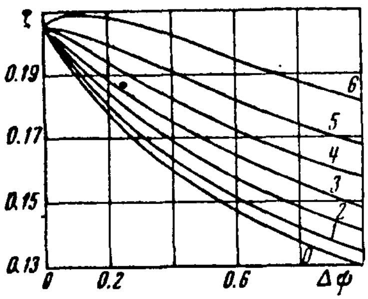

Fig. 2

The difference in the distribution of ζ on ld manifests itself in the shape of the initial section of the optimal nozzle. In Fig. 3 the distribution of ζ on the section ad of the compound nozzle corresponding to Curve 6 (Fig. 2) is given by the solid curve, and the distribution of ζ on the wall of the optimum continuous nozzle having the same final point is shown by the dashed curve.

The optimal compound nozzles were compared with continuous nozzles having length X and optimal for the counterpressure $p_\Sigma^+ = np^+ + n^\circ p^{+\circ}$. We recall that in the class of continuous nozzles such nozzles are optimal also for the problem under consideration. Figure 4 shows one of the optimal compound contours and the contour of the corresponding continuous nozzle (dashed line). It is interesting to note that in all calculated examples the section ad of the compound nozzle turned out (as in the example of Fig. 4) to be close to the initial section of the continuous nozzle that is optimal for counterpressure p_Σ^+.

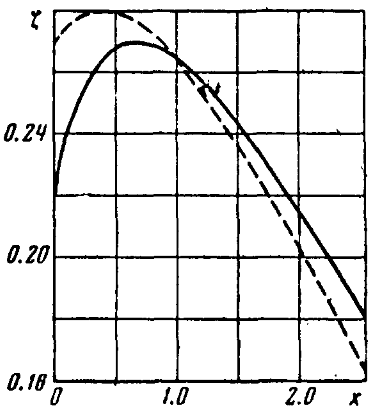

Fig. 3

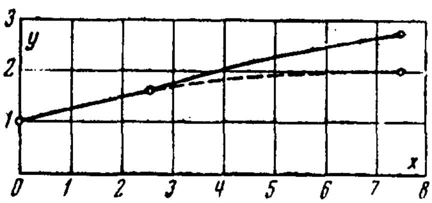

Fig. 4

To estimate the gain that the compound nozzle gives in the case when condition (2.10) is violated, the relative increase $\Delta\chi/\chi_{ab^\circ}$ in the integral of the pressure force was calculated. Here b° is the end point of the continuous nozzle; χ_{ab° is the integral over the section ab° of the contour of the continuous nozzle, analogous to the integrals in (1.3); and $\Delta\chi$ is the difference of χ_Σ and the corresponding value for the continuous nozzle. The values of $\Delta\chi / \chi_{ab^\circ}$ obtained in a series of examples, together with the parameters X, n, p^+ and $p^{+\circ}$, and also some geometric properties are presented in Table II

Table II

X	= 1.72	3.20	3.26	3.30	4.34	7.51
n	= 0.47	0.56	0.53	0.51	0.39	0.49
$p^+ \times 10^2$	= 2.12	0.84	0.30	0.05	0.72	0.44
$p^{+\circ} \times 10$	= 2.66	1.76	1.72	1.70	1.03	0.75
y_b	= 1.45	1.89	1.96	2.00	1.99	2.80
x_d	= 0.58	0.95	0.95	0.95	2.50	2.55
y_d	= 1.11	1.22	1.22	1.22	1.52	1.61
y_{b°	= 1.27	1.55	1.55	1.55	1.69	1.99
$\Delta\chi / \chi_{ab^\circ}$	= 0.05	0.07	0.08	0.07	0.03	0.06

5. The continuous contour, and the configuration shown in Fig. 1 and investigated in the preceding sections, do not exhaust the whole variety of possible shapes for an optimal compound nozzle. This follows from considerations of continuity and comparison of Fig. 1 with Fig. 5a, in which is shown the optimal configuration in the absence of a limitation on the length of the nozzle, that is, for $X = \infty$. In this case the optimal compound nozzle is a combination of two nozzles, each of which provides a uniform stream at the exit. Thus the characteristics ld, de and bg are rectilinear, the gas parameters are constant in the triangle lde , so that $\zeta \equiv 0$ and $p \equiv p^{+\circ}$, and on bg the flow is also parallel to the x-axis and $p \equiv p^{+}$.

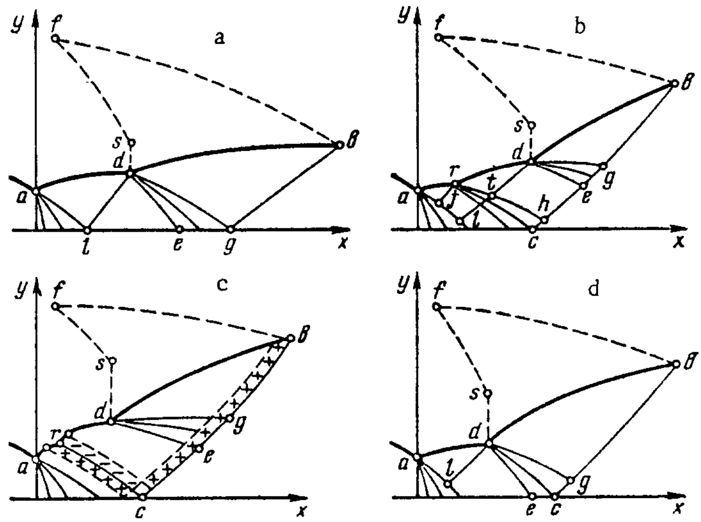

Fig. 5

We denote by X_m the minimum value of X for which the optimal configuration shown in Fig. 5a is possible. This quantity is a function only of p^+ and $p^{+\circ}$ and is obtained if we take as ad and db the contour of minimal length that ensures the constancy of the gas parameters on the characteristics ld and bg. The latter, as is shown in Fig. 5a, have corners at points a and d. We construct the picture of the evolution of the shape of the optimum compound nozzle for increase of X from zero to X_m.

We fix the shape of the nozzle to the left of point a and the values p^+, $p^{+\circ}$ and n. In accord with the condition (2.10) we may expect that the continuous contour achieves the maximum χ_Σ for $X \leqslant X_1$. Here X_1 is the limiting value of X for this case, corresponding to the equality sign in the condition (2.10), and is a function of p^+, $p^{+\circ}$ and n. As soon as X becomes greater than X_1, a final section db appears in the optimal configuration, that is, a compound nozzle is realized of the type already investigated. It can be shown that if the equality is satisfied in (2.10), this ensures the satisfaction of the conditions $A = 0$ and $B = 0$ from (2.9) at $x_d = x_b$. Consequently, the length of the final section tends to zero for $X \to X_1 + 0$, although the corner at point d thereby remains finite. With the growth of $X > X_1$ (if p^+, $p^{+\circ}$ and n are fixed), the initial as well as the final section grows in length.

The type of compound nozzle shown in Fig. 1 achieves the maximum χ_Σ for

$X_1 \leqslant X \leqslant X_2$, where X_2 is the second limiting value of X which, like X_1, is a function of p^+, p^{+o} and n. Values $X_2 < X < X_3$, where $X_3 = X_3(p^+, p^{+o}, n)$ is a third limiting value, correspond to the situation in which point c (the point of intersection of the closing characteristic bc with the axis of symmetry) lies between the last characteristic of the expansion fan arising from point a and the first characteristic of the analogous fan formed by flow past point d. This configuration was not considered above and therefore requires a more detailed analysis.

Since now the boundary of the region of influence includes a segment of the axis of symmetry, where $v \equiv 0$, a boundary condition is required also for μ_1. This condition is obtained just as the other boundary conditions were for the associated problem, and has the form

$$\mu_1 = 0 \quad \text{at} \quad y = 0 \tag{5.1}$$

Further, just as in [17, 18], it can be shown that the characteristic rc of the second family that joins point c with point r on the contour ad, and the characteristic of the first family passing through r are, just like ld, lines of discontinuity of the Lagrange multipliers. On each of these discontinuities one of the equalities (2.4) is satisfied, where the intensity of the jump in μ_1 on rc is determined by the relation

$$[\mu_1] = C(y^\nu \rho\beta)^{1/2} \quad \text{on} \quad rc \tag{5.2}$$

Here the constant C is the same as in equations (2.7), and $[\mu_1]$, just as above, is the difference in the values of μ_1 to the right and left of rc.

It can be shown that the optimal contour adb in the case under consideration has an additional corner at point r, with the flow forming an expansion fan, and consequently the optimal configuration has the form shown in Fig. 5b.

The presence of a corner at point r is proved just as in [17]. In an analogous way are obtained the conditions

$$\begin{gathered}[y^\nu(p + \mu_2\rho u)]_r + \int_{r-}^{r+} \{\mu_1 dv - \mu_2 y^\nu d(\rho u)\} = 0 \\ [\mu_2 y^\nu \rho v]_r - \int_{r-}^{r+} \{\mu_1 du + \mu_2 y^\nu d(\rho v)\} = 0\end{gathered} \tag{5.3}$$

These conditions determine the value of the discontinuity $\Delta\zeta_r = (\zeta_+ - \zeta_-)_r$ and the position of point c on the axis of symmetry within the expansion fan formed by the flow past point r. In (5.3) all quantities are found just as in the calculation of A and B in (2.8).

The conditions of optimality of the segments ar and rd, which in the present case replace the corresponding equalities in (2.11), are written in the form

$$\mu_2 = u + C_3 \quad \text{on } ar, \qquad \mu_2 = u + C_4 \quad \text{on } rd \tag{5.4}$$

where C_3 and C_4 are constants.

The integrals for μ_i, analogous to those valid previously in the triangle ald, now hold in the triangles ajr and rtd. Here in ajr the first equality from (5.4) is used for the second integral, and in the triangle rtd the second one. The optimal distributions of $\zeta = \zeta(\psi)$ on the characteristics jr and td satisfy the equality (2.12) with replacement of the constant C_1 by C_3 and C_4 respectively.

The necessity of a corner at point r can be shown also by using the method that was employed previously in [19] in investigating the flow past a body close to a wedge. Operating in analogous fashion let us assume that the optimal configuration is similar to that shown in Fig. 5b, but without a corner at point r. We vary this contour, leaving it unchanged outside the interval $(y_r - \Delta y) < y < (y_r + \Delta y)$, where Δy is a small positive quantity. Inside this interval we replace the original contour (as shown in Fig. 5c) by two rectilinear segments that intersect the original contour on the boundaries of the interval, and each other at the point $(x_r + \Delta\zeta\Delta y,\ y_r)$, where $\Delta\zeta$ is a positive quantity of the same order as Δy. By linearizing the flow equations with respect to the original (nonuniform) stream, it can be shown that, with an accuracy of higher order than $\Delta\zeta\Delta y$, the perturbations in p induced by the variation carried out on the contour vanish everywhere outside strips of height Δy adjacent to the characteristics rc and bc. Increases (decreases) in pressure correspond to "plus" ("minus") signs in Fig. 5c. It can further be shown that the increment in χ_Σ because of the changes in p on the altered section of the contour is also a quantity of higher order of smallness than $\Delta\zeta\Delta y$. Thus, if $\zeta_h > 0$, which holds in the general case, then there remains only an uncompensated increment in χ_Σ of order $\Delta\zeta\Delta y$, which appears at the expense of an increase of p in the vicinity of point b. Consequently, in contradiction to the assumption made, the original smooth contour is nonoptimal.

There is an interesting mechanism of transition from the optimal configuration of Fig. 1 to the optimal configuration of Fig. 5b which, according to what is said above, takes place at $X = X_2$. In the general case (for $\zeta_h \neq 0$) this transition is realized not by means of "slipping down" of point h to the axis of symmetry along the closing characteristic of the first fan, but as a result of "splitting" of that fan in two. The instant of "splitting" is determined by the position of point h on the closing characteristic of the first fan such that if we introduce the section ar of zero extent, that is divide the fan in two, it is possible to obtain the case shown in Fig. 5b and simultaneously satisfy both conditions (5.3). The fulfillment of one of these conditions at the instant in question takes place at the expense of displacement of point h along the closing characteristic, and the second condition thanks to the choice of the characteristic dividing the splitting fan. For $\zeta_b \neq 0$ the constant C in (5.2) is also different from zero. This together with the relations (2.4) on lines of discontinuity and the condition for μ_1 from (2.5) make impossible the simultaneous satisfaction of the two conditions (5.3) for any other kind of transition. We note that the mechanism of "splitting" described above apparently plays an analogous role in the case of two-phase and nonequilibrium flows, where the contour of the optimal nozzle may also contain an internal corner point [17, 20].

Increase of X from X_2 to X_3 leads to growth in the lengths of the sections ar and db and simultaneously to displacement of point r toward point d. For $X = X_3$ "confluence" takes place of points r and d and of the corresponding fans. In the general case, for the same reason that "splitting" of the first fan occurs, the corner at point r is finite at the instant of "confluence", and point c lies inside the fan.

For $X_3 < X < X_m$ the maximum χ_Σ is achieved by the configuration shown in Fig. 5d. In this case the condition (5.1) is fulfilled on the axis of symmetry, and the intensity of the discontinuity in μ_1 on the characteristic dc is given by Eq. (5.2). In other respects the construction of the optimal contour is carried out here just as for the configuration shown in Fig. 1. For $X \to X_m - 0$ there occurs a natural transition to the case of two nozzles with parallel discharge (Fig. 5a). If $0 < n < 1$, then "smoothing" of the streams at the exist of both nozzles apparently takes place such that $y_l > 0$ and $y_g > 0$ for $X < X_m$ and $y_l = y_g = 0$ only for $X = X_m$. In the latter case (for $X = X_m$) the solution of the associated problem is given by the equations

$$\mu_1 = y^\nu \rho v, \quad \mu_2 = u - (1 - n) u_l - n u_g \quad \text{in } alda$$
$$\mu_1 = y^\nu \rho v n, \quad \mu_2 = n(u - u_g) \quad \text{in } bgldb \tag{5.5}$$

In the case of combination of two nozzles with parallel discharge, given continuous distributions of μ_i (for any n) ensure the fulfillment of all the equations and conditions of the variational problem (including $A = B = 0$). It is possible to convince oneself of this by direct substitution of (5.5) into the indicated equations and conditions.

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ON ONE TYPE OF INTERACTION OF THE BOUNDARY LAYER AND THE OUTER (INVISCID) STREAM AT SUPERSONIC SPEEDS

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The problem is considered of unsymmetric steady flow past a circular cone in a uniform supersonic stream of viscous gas at high Reynolds number R. It was shown in [1] that in many cases the solution of the problem of inviscid flow past a cone is such that normal derivatives of the density (and temperature) and of the velocity components of the gas tangent to the surface become infinite at the surface of the cone. In these cases, it follows from the condition of matching the solution for inviscid flow past the cone (which is regarded as the first term of an asymptotic expansion of the solution of the complete problem in powers of $\varepsilon = R^{-1/2}$ outside the boundary layer) with the solution of the problem in the boundary layer that supplementary terms appear in the latter solution, which may give a significant correction to the results of the usual boundary-layer theory. It is shown (in the case of a laminar boundary layer) that these supplementary terms are self-similar; and a strict formulation is given of the problem for their determination.

1. We consider steady flow past a circular cone of semi-vertex angle β in a uniform supersonic stream of viscous gas at angle of attack α. In a system of coordinates in